Feb 07, 2017

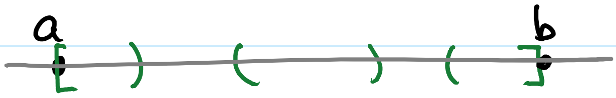
Thursday, February 9, 2017 9:18 AM

Given a topological space $(X, \mathcal{T})$ and $A \subset X$. How to define a topology for A which is coming from X?

Definition. $\mathcal{T}|_A = \{ G \cap A : G \in \mathcal{T} \}$ is called the subspace topology, or relative topology, or relative topology on A from X.

Exercise. One has to check the conditions that $\mathcal{T}|_A$ is really a topology.

Example. $(X, \mathcal{T}) = (\mathbb{R}, \text{std})$ and $A = [a,b]$



$[a, a+\varepsilon)$ (c, d) $(b-\delta, b]$

are typical open sets in $\mathcal{T}|_{[a,b]}$

They form a base.

Example. $(X, \mathcal{T}) = (\mathbb{R}, \text{std})$, $A = (1,2) \cup [3,4)$

Think about these questions.

* Which one is an open set in A?
  $(3,4)$ or $[3,4)$
* Is the set $(1,2)$ closed in A?

Question. What are the induced topology on $\mathbb{Q}$ from $(\mathbb{R}, \text{std})$ or $(\mathbb{R}, \text{lower-limit})$?

02-07-p2

Thursday, February 9, 2017 9:19 AM

From these example, we natural observe that for $P \subset A \subset X$ in $(X, \mathcal{J})$

$$P \in \mathcal{J} \underset{\longleftarrow}{\Longrightarrow} P \in \mathcal{J}|_A$$

condition

The condition is clearly $A \in \mathcal{J}$, that is, if $A \in \mathcal{J}$ then $P \in \mathcal{J} \Leftrightarrow P \in \mathcal{J}|_A$.

Easy exercise. Is the converse true?

Exercise. Is it true that

$$\mathcal{J}|_P = (\mathcal{J}|_A)|_P \ ?$$

Given $(X, \mathcal{J})$ and $A \subset X$ and

$$f : X \longrightarrow Y, \text{ we also have}$$

$$f|_A : A \longrightarrow Y$$

Naturally, one would expect

$$f \text{ is continuous} \Longrightarrow f|_A \text{ is continuous}$$

The proof is simply as below.

Let $V \in \mathcal{J}_Y$ and we need to consider

$$(f|_A)^{-1}(V) = \underbrace{f^{-1}(V)}_{\text{in } \mathcal{J}} \cap A \in \mathcal{J}|_A \text{ by definition}$$

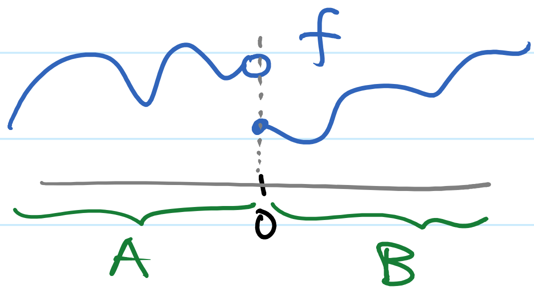
Key consideration.

If $f|_A$ is continuous on many A's $\subset X$, how to conclude f is continuous on X.

Bad example. $(X, \mathcal{T}) = (\mathbb{R}, \text{std})$

Write $\mathbb{R} = \underbrace{(-\infty, 0)}_{A} \cup \underbrace{[0, \infty)}_{B}$

This $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies both $f|_A, f|_B$ are continuous but f is not



Proposition. Given $(X, \mathcal{T})$, $f : X \longrightarrow Y$ and

$X = \bigcup_\alpha G_\alpha$ where each $G_\alpha \in \mathcal{T}$

If each $f|_{G_\alpha} : G_\alpha \longrightarrow Y$ is continuous then so is $f : X \longrightarrow Y$

Proof. Take arbitrary $V \in \mathcal{T}_Y$.

$$f^{-1}(V) = f^{-1}(V) \cap \left(\bigcup_\alpha G_\alpha\right) = \bigcup_\alpha \left[f^{-1}(V) \cap G_\alpha\right]$$

$$= \bigcup_\alpha \underbrace{(f|_{G_\alpha})^{-1}(V)}_{\text{open in } G_\alpha,\ \therefore \text{ open in } X}$$

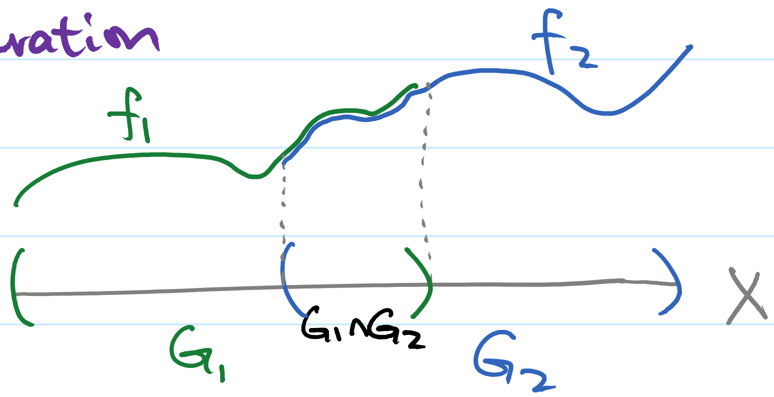
Hence $f^{-1}(V)$ is a union of open sets in $\mathcal{T}$

Another verion. Given $X = \bigcup_\alpha G_\alpha$ as above.

If we have a family of continuous mappings $f_\alpha : G_\alpha \longrightarrow Y$ satisfying $f_\alpha \equiv f_\beta$ on $G_\alpha \cap G_\beta$ then $\exists$ continuous $f : X \longrightarrow Y$ such that

$$f|_{G_\alpha} \equiv f_\alpha$$

An illustration



Then a continuous f can be defined on X

Think. Compare this with previous bad example.
The "bad" becomes "good" on $(\mathbb{R}, \text{lower-limit})$.

Question. We need $G_\alpha \in \mathcal{J}$ in the above, can it be changed or relaxed?

Proposition. Let $X = A \cup B$ where A, B are closed. If $f : X \longrightarrow Y$ satisfies that both
$f|_A, f|_B$ are continuous
then so is f.

Proof The simplest one should involve an equivalent version ⑥ of continuity.
Take a closed $H \subset Y$ and consider

$$f^{-1}(H) = (f|_A)^{-1}(H) \cup (f|_B)^{-1}(H).$$

Remark. Obviously, for closed sets, only finitely many are allowed.

02-07-p5

Thursday, February 9, 2017 9:20 AM

Uniqueness Theorem. Given X and a Hausdorff Y, $A \subset X$ where A is dense, and continuous functions $f, g : X \longrightarrow Y$.

If $f|_A \equiv g|_A$ then $f \equiv g$ on X.

Remark. This theorem tells us that two f, g are indeed the same one. It cannot be use to answer existence problem, e.g. can we find a continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(p/q) = 1/q \text{ for each } p/q \in \mathbb{Q}.$$

Proof. Need to prove $f(x) = g(x) \in Y \ \forall x \in X$

Observe that Hausdorff property tells us what happens for $y_1 \neq y_2$ in Y

So, we start by negation and look for contradiction.

Suppose $\exists x \in X$, $f(x) \neq g(x)$. Then

$\exists V_1, V_2 \in \mathcal{T}_Y$, $f(x) \in V_1$, $g(x) \in V_2$, $V_1 \cap V_2 = \phi$

Then $x \in f^{-1}(V_1) \in \mathcal{T}_X$, $x \in g^{-1}(V_2) \in \mathcal{T}_X$

$\therefore x \in f^{-1}(V_1) \cap g^{-1}(V_2) \in \mathcal{T}_X$

Since $\bar{A} = X$, $f^{-1}(V_1) \cap g^{-1}(V_2) \cap A \neq \phi$

$\therefore \exists \underset{\cup}{a}$, but $f(a) = g(a)$

$\therefore \quad f(a) = g(a) \in V_1 \cap V_2 = \phi$ contradiction

02-07-p6

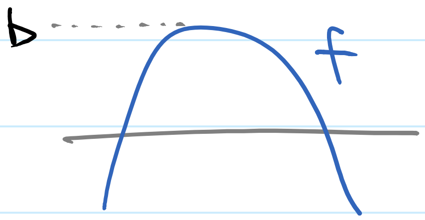
Thursday, February 9, 2017 9:20 AM

Definition. $f: X \longrightarrow Y$ is called

* **homeomorphism** if f is a bijection and both f, f^{-1} are continuous
* **open mapping** if $\forall\, U \in \mathcal{T}_X$, $f(U) \in \mathcal{T}_Y$

Remark. A homeomorphism can be re-stated as a bijection which is both open and continuous.

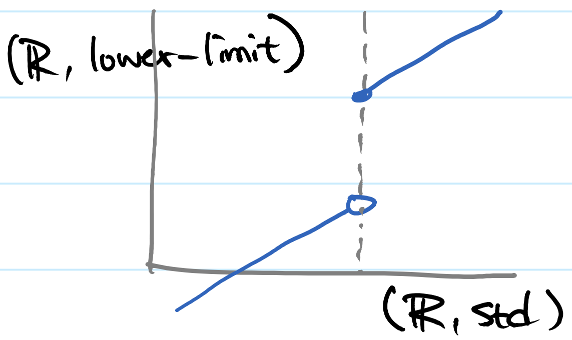
Example. Continuous but not open



$f(\mathbb{R}) = (-\infty, b]$

open   not open

Example. open but not continuous



Exercise. Verify this example.

Example. Open and continuous but not homeomorphism

$$(\mathbb{R}, \text{std}) \longrightarrow S^1 \subset (\mathbb{C} = \mathbb{R}^2, \text{std})$$

$$x \longmapsto e^{2\pi i x}$$